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Integrability Conditions for the Fourier Transform

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I. INTRODUCTION

In this paper, we prove a simple integrability condition for Fourier transforms of functions in $L(p, q)$ and for functions in $L^q_{|t|^{1/p-1/q}}$. With the help of this condition we prove a strong version of a conjecture of Boas [1] concerning integrability of Fourier transforms of monotone functions, and of functions whose Fourier transforms are monotone.

Our condition has a natural analog for periodic functions, which in turn provides a new proof for a classical theorem of Hardy and Littlewood on functions whose Fourier coefficients are monotone.

We will review in brief the definitions of the spaces used: If (X, Σ, μ) is a σ-finite measure space, define:

$$f_*(y) = \mu\{x \mid |f(x)| > y\}, f^*(t) = \operatorname{Inf}\{y \mid f_*(y) \leqslant t\}.$$

For $0 < p < \infty$, $0 < q < \infty$ define:

$$|f|^*_{p,q} = \left(\int_0^\infty f^*(t)^q t^{q/p} \frac{dt}{t}\right)^{1/q},$$

while for $0 < p \leqslant \infty$, $q = \infty$

$$|f|^*_{p,\infty} = \sup_{0<t} t^{1/p} f^*(t).$$

Define $L(p, q) = \{f \mid |f|^*_{p,q} < \infty\}$. Given $0 \leqslant \omega(x)$, measurable, we define $L_\omega{}^p = \{f \mid \omega f \in L^p\}$, with the obvious norm (we will have only $p \geqslant 1$). Denote for typographical convenience:

$$|t|^{1/p-1/q} = t(p, q).$$

Denote by E the class of nonnegative, even functions on R, monotone decreasing to 0 on R^+. Denote by $\mathcal{O}$ the class of functions f such that $\pm\text{sign } x \cdot f(x) \in E$. Boas' conjecture is: For $f \in E$ (or $f \in \mathcal{O}$) $f \in L^q_{t(p,q)}$ iff

$\hat{f} \in L^q_{t(p',q)}$, where $1/p + 1/p' = 1$, $1 < p < \infty$. Given f on R, denote by $m(f)$its least majorant in E:

$$m(f)(x) = \sup\{|f(t)| \mid |t| \geqslant |x|\}.$$

Now $f \in L(p, q)$ iff $f^* \in L^q_{t(p,q)}$ so that sometime $L(p, q) \supset L^q_{t(p,q)}$, and sometime $L(p, q) \subset L^q_{t(p,q)}$, depending on whether $p \leqslant q$ or $q \leqslant p$. The spaces coincide of course for $f \in E$ or $f \in \mathcal{O}$. We shall show: For $f \in E$ or $f \in \mathcal{O}$,

$$f \in L^q_{t(p,q)} \Rightarrow m(\hat{f}) \in L^q_{t(p',q)} \Rightarrow \hat{f} \in L^q_{t(p',q)} \Rightarrow f \in L^q_{t(p,q)};$$

as well as,

$$f \in L(p, q) \Rightarrow m(\hat{f}) \in L(p', q) \Rightarrow \hat{f} \in L(p', q) \Rightarrow f \in L(p, q).$$

We shall need several results from interpolation theory. We give a short outline of the necessary notions, not in their most general form, and refer the reader to [2–4] for details.

Let A_0, A_1 be quasi-normed commutative semigroups(i.e., $+ : A_i \times A_i \to A_i$ is associative, commutative and A_i posseses a 0. Further, $|\ |_i: A_i \to R^+$ satisfies $|a|_i = 0 \Leftrightarrow a = 0$ and $|a_0 + a_1|_i \leqslant |a_0|_i + |a_1|_i$). Assume $A_i \subset \mathcal{A}$, $\mathcal{A}$ a topological vector space and $|a_n|_i \to 0$ implies $a_n \to 0$ in $\mathcal{A}$. For $0 < \theta < 1$, $0 < q \leqslant \infty$ we construct new semigroups $(A_0, A_1)_{\theta,q}$. The construction has the following property: If $B_0, B_1 \subset \mathcal{B}$ is second interpolation pair, and if $T: A_i \to B_i$ is bounded and satisfies a mild linearity condition (satisfied by all the operators that will occur later). Then $T: (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$ is bounded. The construction also gives:

(1) $(L(p_0, q_0), L(p_1, q_1))_{\theta,q} = L(p, q)$ where $1/p = ((1 - \theta)/p_0) + (\theta/p_1)$; $0 < q \leqslant \infty$ arbitrary.

(2) $(L^p_{\omega_0}, L^p_{\omega_1})_{\theta,p} = L^p_{\omega_0^{1-\theta}\omega_1^{\theta}}$ (Stein–Weiss).

(3) E, $\mathcal{O}$ have the Marcinkiewicz property in the $L(p, q)$ scale:

$$(E \cap L(p_0, q_0), E \cap L(p_1, q_1))_{\theta,q} = E \cap (L(p_0, q_0), L(p_1, q_1))_{\theta,q},$$

and similarly for $\mathcal{O}$ (see [3]).

Finally, a notation convention: All integrals without limits of integration are on $(-\infty, \infty)$.

II. $L(p,q)$ Conditions

Let $\mathcal{S}$ denote the class of $C^\infty(R)$ functions all of whose derivatives decrease rapidly to 0 at $\pm\infty$. $f \in L(p, q)$ implies $\hat{f} \in \mathcal{S}'$, the space of tempered distributions. Therefore, we can consider, for any $\varphi \in \mathscr{S}$, the following function of t:

$$(\varphi_t, \hat{f}) = \phi(t),$$

where $\varphi_t(u) = (1/t)\,\varphi(u/t)$, and $\hat{f}$ acts on the u variable.

THEOREM 2.1. $|\,m(\phi)|^*_{p',q} \leqslant C(p, \varphi)\,|\,f\,|^*_{p,q}$, *where* $1 < p < \infty$; $0 < q \leqslant \infty$.

Proof. $(\varphi_t, \hat{f}) = \operatorname{sign} t(\hat{\varphi}(tu), f(u)) = (\int f(u)\,\hat{\varphi}(tu)\,du) \operatorname{sign} t$. Therefore,

$$|(\varphi_t, \hat{f})| \leqslant |\,f\,|_p \left(\int |\,\hat{\varphi}(tu)|^{p'}\,du\right)^{1/p'} = |\,f\,|_p\,|\,\hat{\varphi}\,|_{p'} \cdot |\,t\,|^{-1/p'}.$$

so that $m(\phi)(t) \leqslant |\,f\,|_p\,|\,\hat{\varphi}\,|_{p'}\,|\,t\,|^{-1/p'}$. Consider the sublinear operator $T: f \to m(\phi)$. We have shown, for $1 \leqslant p \leqslant \infty$,

$$T: L(p, p) \to L(p', \infty).$$

Interpolating, we get

$$T: L(p, q) \to L(p', q).$$

THEOREM 2.2. $f \in L(p, q)$ *and* $\hat{f} \in E$, *then* $\hat{f} \in L(p', q)$ $1 < p < \infty$, $0 < q \leqslant \infty$.

Proof. Take $\varphi \in \mathscr{S}$ nonnegative, with support in $[-1, 1]$. We then have:

$$\operatorname{sign} t(\varphi_t, \hat{f}) = \frac{1}{|\,t\,|} \int \varphi\left(\frac{u}{t}\right) \hat{f}(u)\,du \geqslant \hat{f}(t)\,\frac{1}{|\,t\,|} \int \varphi\left(\frac{u}{t}\right) du = \hat{f}(t) \cdot |\,\varphi\,|_1 .$$

Therefore,

$$0 \leqslant \hat{f} \leqslant \frac{1}{|\,\varphi\,|}\,\phi, \qquad \text{and so} \qquad |\,\hat{f}\,|^*_{p',q} \leqslant \frac{1}{|\,\varphi\,|}\,|\,\phi\,|^*_{p',q} \leqslant C(p)\,|\,f\,|^*_{p,q} .$$

However, note that we have used in the proof much less than $\hat{f} \in E$. In fact, all we used was

$$0 \leqslant \hat{f}(t) \leqslant \frac{C}{|\,\varphi\,|_1}\,\frac{1}{|\,t\,|} \int \hat{f}(u)\,\varphi\left(\frac{u}{t}\right) du,$$

for a nonnegative $\varphi \in \mathscr{S}$. Thus, $\hat{f}(t)$ may vanish on stretches, and still the conclusion may hold.

THEOREM 2.3. $f \in L(p, q) \cap E$, *then*

$$|\,m(\hat{f})|^*_{p',q} \leqslant C_p\,|\,f\,|^*_{p,q}, \qquad 1 < p < \infty; \quad 0 < q \leqslant \infty.$$

Proof. Assume first $f \in L(1, 1) \cap E$,

$$\hat{f}(t) = \int f(x)\, e^{ixt}\, dx = -\frac{1}{it} \int e^{ixt}\, df(t).$$

Therefore, $|\hat{f}(t)| \leqslant 2f(0)/|t| = 2\,|f|_\infty/|t|$, so that also $m(\hat{f}) \leqslant 2\,|f|_\infty/t$. $L(1, 1) \cap E$ is dense in $L(\infty, \infty) \cap E$, so that we have, for $Tf = m(\hat{f})$,

$$T: L(\infty, \infty) \cap E \to L(1, \infty),$$

since also

$$T: L(1, 1) \to L(\infty, \infty),$$

and since E has the Marcinkiewicz property in the $L(p, q)$ scale,

$$T: L(p, q) \cap E \to L(p', q).$$

Theorems 2.2 and 2.3 prove for $f \in E$:

$$f \in L(p, q) \Rightarrow m(\hat{f}) \in L(p', q) \Rightarrow \hat{f} \in L(p', q) \Rightarrow f \in L(p, q).$$

The proofs for $f \in \mathcal{O}$ are the same. The analog of Theorem 2.2, for periodic functions is easier to state for functions with period π.

THEOREM 2.4. $f \in L(p, q)$. *Denote*:

$$\sigma_n = (1/n)(\hat{f}(1) + \cdots + \hat{f}(n)).$$

Then $|\{m(\sigma_n)\}|_{p',q} \leqslant C_p\, |f|_{p,q} \cdot 1 < p < \infty;\ \ 0 < q \leqslant \infty.$

Proof. Do the sine-transform:

$$\sigma_n = \int_0^\pi f(t)\, \frac{\cos t/2 - \cos(n + \frac{1}{2})\, t}{2n \sin t/2}\, dt,$$

$$|\sigma_n| \leqslant |f|_p \left(\int_0^\pi \left| \frac{\cos t/2 - \cos(n + \frac{1}{2})\, t}{2n \sin t/2} \right|^{p'} dt\right)^{1/p'}.$$

Since in $(0, \pi)$ $\sin t/2 \geqslant (1/\pi)t$,

$$\left(\int_0^\pi \left| \frac{\cos t/2 - \cos(n + \frac{1}{2})\, t}{2n \sin t/2} \right|^{p'} dt\right)^{1/p'}$$

$$\leqslant \frac{\pi}{2} \left(\int_0^\pi \left| \frac{1 - \cos nt}{nt} + \frac{1}{n} \right|^{p'} dt\right)^{1/p'}$$

$$\leqslant \frac{\pi^2}{2n} + \frac{\pi}{2}\, n^{-1/p'} \left(\int_0^\infty \left(\frac{1 - \cos t}{t}\right)^{p'} dt\right)^{1/p'} \leqslant C_p n^{-1/p'}.$$

Therefore, $m(\sigma_n) \leqslant |f|_p C_p n^{-1/p'}$. $Tf = m(\sigma_n)$ satisfies

$$T: L(p, q) \to l(p', \infty).$$

Interpolating, we get

$$T: (p, q) \to l(p', q), \qquad 1 < p < \infty; \quad 0 < q \leqslant \infty,$$

and the theorem is proved.

The proof of the Hardy–Littlewood theorem using Theorem 2.4, is now straightforward.

III. Weighted L^q Conditions

THEOREM 3.1. *If $f \in L^q_{t(p,q)}$, $1 < p < \infty$, we have $f \in \mathscr{S}'$, and define as before for $\varphi \in \mathscr{S}$, $\phi(t) = (\varphi_t, \hat{f})$. We have:*

$$| m(\phi)|^*_{p',q} \leqslant C(p, \varphi) \, |f|_{L^p_{t(p,q)}}.$$

Proof. As before, sign $t\phi(t) = \int \hat{\varphi}(ut) f(u)\, du$, so that

$$|\phi(t)| \leqslant |f|_{L^q_{u(p,q)}} |\hat{\varphi}|_{L^{q'}_{v(p',q')}} |t|^{-1/p'}.$$

Therefore, $T: f \to m(\phi)$ satisfies

$$T: L^q_{u(p,q)} \to L(p', \infty).$$

Interpolating between different values of p for fixed q, and using the Stein–Weiss theorem for interpolation with change of measure, we get

$$T: L^q_{u(p,q)} \to L(p', q).$$

The theorem is proved.

THEOREM 3.2. $f \in L^q_{t(p,q)}$, $\hat{f} \in E$ *then* $\hat{f} \in L^q_{t(p',q)}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$.

Proof. Same as of Theorem 2.2.

Since for $f \in E$, $f \in L^q_{t(p,q)} \Leftrightarrow f \in L(p, q)$. We have: If $f \in E$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$:

$$f \in L^q_{t(p,q)} \Rightarrow m(\hat{f}) \in L^q_{t(p',q)} \Rightarrow \hat{f} \in L^q_{t(p',q)} \Rightarrow f \in L^q_{t(p,q)}.$$

The proof of the case $f \in \mathscr{O}$ is similar and we have proved the strong version of Boas' conjecture. As in II, we can prove the periodic analog of 3.1. We leave this to the reader.

References

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